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EXTENSION OF GENERAL CLASS OF GENERATING FUNCTIONS AND ITS APPLICATIONS-I

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Abstract: In this paper, we introduce a general class of generating functions involving the triple product of modified Laguerre polynomials $L_n^{(\alpha-n)}(x)$, modified Jacobi polynomials $P_m^{(\alpha,\beta-m)}(q)$ and the confluent hypergeometric functions ${}_1F_1[.]$ and then obtain its some more general class of generating functions by group-theoretic approach and discuss their applications. Earlier Bhandari [1] introduce a general class of generating functions involving the product of modified Jacobi polynomials $P_n^{(\alpha,\beta-n)}(x)$ and the confluent hypergeometric functions ${}_1F_1[.]$.

Keywords and Phrases: Generating functions, Modified Laguerre polynomials, Modified Jacobi polynomials, Confluent hypergeometric functions.

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1. Introduction

The modified Laguerre polynomials $L_n^{(\alpha-n)}(x)$ and modified Jacobi polynomials $P_m^{(\alpha,\beta-m)}(q)$ are defined by Srivastava and Manocha [5] as:.

$$L_n^{(\alpha-n)}(x) = \frac{\Gamma(1+\alpha)}{\Gamma(1+n)\Gamma(1+\alpha-n)}\ {}_1F_1\left[-n; 1+\alpha-n; x\right] \tag{1.1}$$

$$P_m^{(\alpha,\beta-m)}(q) = \frac{(1+\alpha)_m}{m!}\ {}_2F_1\left[-m, 1+\alpha+\beta+m; 1+\alpha; \frac{1-q}{2}\right] \tag{1.2}$$

The confluent hypergeometric functions ${}_1F_1[.]$ can be replaced by many special functions such as the Bessel polynomials. Srivastava and Manocha [5] defined and studied various bilinear, bilateral and multilinear generating functions.
In this paper, we introduce the following new general class of generating functions:

$$G(x,q,u,w)=\sum_{n=0}^{\infty}a_n L_n^{(\alpha-n)}(x)P_m^{(\alpha,\beta-m)}(q)\ {}_1F_1\left[-n;m+1;u\right]w^n \tag{1.3}$$

where a_n is any arbitrary sequence independent of x,q,u and w.
Again in (1.3) setting various values of a_n, we may find several results on generating functions involving different special functions, hence (1.3) is a general class of generating functions. In this paper, we evaluate some more general class of generating functions and finally discuss their applications.

2. Group Theoretic Operators

In our investigations, we use the following group-theoretic operators:
The operators R_1 due to Majumdar [3] is given by

$$R_1=xyz\frac{\partial}{\partial x}-y^2z\frac{\partial}{\partial y}-(x-\alpha)yz \tag{2.1}$$

Such that

$$R_1\left[L_n^{(\alpha-n)}(x)y^nz^\alpha\right]=(n+1)L_{n+1}^{(\alpha-n-1)}(x)y^{n+1}z^{\alpha+1} \tag{2.2}$$

The operators R_2 due to Chongdar [2] is given by

$$R_2=(1-q^2)r\frac{\partial}{\partial q}-2r^2\frac{\partial}{\partial r}-\left[(1+\alpha+\beta+p)(1+q)-2\beta\right]r \tag{2.3}$$

Such that

$$R_2\left[P_m^{(\alpha,\beta-m)}(q)r^m\right]=-2(m+1)P_{m+1}^{(\alpha,\beta-m-1)}(q)r^{m+1} \tag{2.4}$$

The operator R_3 due to Miller Jr. [4] is given by

$$R_3=v\frac{\partial}{\partial t}+vut^{-1}\frac{\partial}{\partial u}-vut^{-1} \tag{2.5}$$

Such that

$$R_3\left[\ {}_1F_1\left[-n;m+1;u\right]v^nt^m\right]=m\ {}_1F_1\left[-n-1;m;u\right]v^{n+1}t^{m-1} \tag{2.6}$$

The actions of R_1, R_2 and R_3 on function f are obtained as follows:

$$e^{wR_1}f(x,y,z)=(1+wyz)^\alpha\exp(-wxyz)f\left[x(1+wyz),\frac{y}{1+wyz},z\right] \tag{2.7}$$

[Majumdar [3]]

$$e^{wR_2} f(q,r) = \{1 + wr(1+q)\}^{-1-\alpha-\beta-p} (1+2wr)^{\beta} \ F\left[\frac{q + wr(1+q)}{1 + wr(1+q)}, \frac{r}{1+2wr}\right] \tag{2.8}$$

[Chongdar [2]]

and

$$e^{wR_3} f(v,t,u) = \exp\left(\frac{-uvw}{t}\right) f\left[v, t+wv, u\left(1+\frac{wv}{t}\right)\right] \tag{2.9}$$

[Millar Jr. [4]]

3. Some more General class of Generating Functions

In this sections, making an use of the general class of generating function (1.3) and group-theoretic operators R_1, R_2 and R_3 with their actions given in the section 2, we obtain some more general class of generating functions through following theorem:

Theorem 3.1. If there exists a general class of generating functions involving the triple product of modified Laguerre polynomials $L_n^{(\alpha-n)}(x)$, modified Jacobi polynomials $P_m^{(\alpha,\beta-m)}(q)$ and the confluent hypergeometric functions ${}_1F_1\left[-n; m+1; u\right]$ given by

$$G(x,q,u,w) = \sum_{n=0}^{\infty} a_n L_n^{(\alpha-n)}(x) P_m^{(\alpha,\beta-m)}(q) \ {}_1F_1\left[-n; m+1; u\right] w^n \tag{3.1}$$

Then the following more general class of generating functions holds:

$$(1+w)^m (1+2wr)^{\beta-m} (1+w)^{\alpha} \{1+wr(1+q)\}^{-1-\alpha-\beta-p} \exp\left[-w(x+u)\right].$$

$$G\left[x(1+w), \frac{q+wr(1+q)}{1+wr(1+q)}, u(1+w), \frac{wyv}{1+w}\right] = \sum_{n,i,j,k=0}^{\infty} \frac{a_n (-2)^i (n+1)_i (m+1)_j}{i!j!k!}.$$

$$L_{n+i}^{(\alpha-n-i)}(x) P_{m+j}^{(\alpha,\beta-m-j)}(q) \ {}_1F_1[-n-k; m-k+1; u] w^i (wr)^j (mw)^k (wyv)^n \tag{3.2}$$

Proof. In the general class of generating functions (3.1), replacing w by wyv and then multiplying by $z^{\alpha} r^m t^m$ on both sides, we get

$$G(x,q,u,wyv) z^{\alpha} r^m t^m$$

$$= \sum_{n=0}^{\infty} a_n L_n^{(\alpha-n)}(x) P_m^{(\alpha,\beta-m)}(q)\ {}_1F_1[-n; m+1; u] y^n v^n . z^\alpha r^m t^m . w^n \tag{3.3}$$

Now, operating both the sides of (3.3) with $e^{wR_1}e^{wR_2}e^{wR_3}$, we obtain

$$e^{wR_1}e^{wR_2}e^{wR_3}[G(x,q,u,wyv)z^\alpha r^m t^m]$$

$$= e^{wR_1}e^{wR_2}e^{wR_3} \sum_{n=0}^{\infty} a_n L_n^{(\alpha-n)}(x) y^n z^\alpha . P_m^{(\alpha,\beta-m)}(q) r^m .\ {}_1F_1[-n; m+1; u] v^n t^m . w^n \tag{3.4}$$

The left hand side of (3.4) becomes

$$z^\alpha (1+wyz)^\alpha \{1+wr(1+q)\}^{-1-\alpha-\beta-p} \left(\frac{r}{1+2rw}\right)^m (1+2wr)^\beta (t+wv)^m$$

$$\exp\left(-w(xyz + \frac{-uv}{t})\right) G\left[x(1+wyz) + \frac{q+wr(1+q)}{1+wr(1+q)}, u\left(1+\frac{wv}{t}, \frac{wyv}{1+wyz}\right)\right] \tag{3.5}$$

and the right hand side of (3.4) becomes

$$\sum_{n,i,j,k=0}^{\infty} \frac{a_n (-2)^i (n+i)_i (m+1)_j m^k w^{n+i+j+k}}{i!j!k!} L_{n+i}^{(\alpha-n-j)}(x) y^{n+i} z^{\alpha+i} P_{m+j}^{(\alpha,\beta-m-j)}(q) r^{m+j}$$

$${}_1F_1[-n-k; m-k+1; u] v^{n+k} t^{m-k} \tag{3.6}$$

Now equating (3.5) and (3.6), and setting $t = v$ and $yz = 1$,

$$(1+w)^\alpha \{1+wr(1+q)\}^{-1-\alpha-\beta-p} (1+2wr)^{\beta-m} (1+w)^m \exp(-wx+u)$$

$$.G\left[x(1+w) + \frac{q+wr(1+q)}{1+wr(1+q)}, u(1+w), \frac{wyv}{1+wyz}\right]$$

$$= \sum_{n,i,j,k=0}^{\infty} \frac{a_n (-2)^i (n+1)_i (m+1)_j}{i!j!k!} L_{n+i}^{\alpha-n-j}(x) P_{m+j}^{(\alpha,\beta-m-j)}(q)$$

$$.{}_1F_1[-n-k; m-k+1; u] w^i (wr)^j (mw)^k (wyv)^n \tag{3.7}$$

4. Special Case

Taking $q = 0, u = 0$ in given theorem and proceeding as the proof of the main theorem, we get

$$\exp(-wx) G\left[x(1+w), \frac{wy}{1+w}\right] = \sum_{n=0}^{\infty}\sum_{i=0}^{\infty} \frac{a_n (n+1)_i w^{n+i}}{i!} L_{n+i}^{(\alpha-n-i)}(x) y^{n+i} z^i$$

$$= \sum_{n=0}^{\infty} \sum_{i=0}^{\infty} \frac{a_{n-i}(n-i+1)_i w^n}{i!} L_n^{(\alpha-n)}(x) y^n z^i$$

$$= \sum_{n=0}^{\infty} \sigma_n(x,z).(wy)^n \tag{4.1}$$

where

$$\sigma_n(x,z) = \sum_{i=0}^{n} \frac{a_{n-i}(n-i+1)_i}{p!} L_n^{(\alpha-n)}(x) z^i \tag{4.2}$$

which is given by Majumdar [3].

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